Math 255A' Lecture 5 Notes

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1 The Hahn-Banach Theorem

1.1 Examples of Dual Spaces

Here are examples of concrete descriptions of some dual spaces.

Example 1.1. If (X, Σ, μ) is a measure space, $1 < p < \infty$, and $p^{-1} + q^{-1} = 1$, then the map $L^q \to (L^p(\mu))^*$ given by $g \mapsto L_g$ is a linear isometry, where $L_g(f) = \int_X fg \, d\mu$.

Example 1.2. If (X, Σ, μ) is σ-finite, then $L^\infty(\mu) \mapsto (L^1(\mu))^*$ given by $g \mapsto L_g$ is an isometric isomorphism, where $L_g(f) = \int fg \, d\mu$.

Example 1.3. Let X be a locally compact Hausdorff space, and let $M(X)$ be the set of $\mathbb{F}$-valued regular[1] Borel measures on X with $\|\mu\|$ equalling the total variation of μ. Then the map $M(X) \to C_0(X)^*$ given by $\mu \mapsto L_\mu$ is an isometric isomorphism, where $M_\mu(f) = \int_X f \, d\mu$.

1.2 The Hahn-Banach theorem

Let X be a vector space over $\mathbb{F}$.

Definition 1.1. A **sublinear functional** on X is a function $p : X \to \mathbb{R}$ such that

1. $p(x + y) \le p(x) + p(y)$
2. $p(\alpha y) = \alpha p(y)$ for all $\alpha \in [0, \infty)$.

Example 1.4. Any seminorm is a sublinear functional.

Theorem 1.1 (Hahn-Banach)**.** *Let $\mathbb{F} = \mathbb{R}$, let M be a linear subspace of X, and let p be a sublinear functional on X. If $f : M \to \mathbb{R}$ is linear and $f \le p|_M$, then there is a linear $F : X \to \mathbb{R}$ such that $F|_M = f$ and $F \le p$.*

[1] For general locally compact spaces, "regular" can have different meanings. Take it to have the meaning that makes this theorem work.

Proof. Step 1: Assume $\dim(X/M) = 1$. Then there is some $x_0 \in X$ such that $M + \mathbb{R}x_0 = X$. We must find something of the form $F(y + tx_0) = f(y) + t\alpha_0$ for some $\alpha_0 \in \mathbb{R}$ such that $F \leq p$. What must α_0 satisfy? We need $f(y) + t\alpha_0 \leq p(y) + tx_0)$ for all $y \in M, t \in \mathbb{R}$.

- If $t > 0$, divide by t to get $f(y') + \alpha_0 \leq p(y' + x_0)$ for all $y' \in M$. That is, we need $\alpha_0 \leq \inf_{y' \in M} p(y' + x_0) - f(y')$.
- If $t < 0$, divide by $-t$ to get $f(y') - \alpha_0 \leq p(y' - x_0)$ for all $y' \in M$. That is, we need $\alpha_0 \geq \sup_{y' \in M} f(y') - p(y' - x_0)$.

It remains to check that for any $y', y'' \in M$, $f(y'') = p(y'' - x_0) \leq p(y' - x_0) - f(y')$ (so such an α_0 exists). We can rearrange this to get $f(y' + y'') \leq p(t' + x_0) + p(y'' - x_0)$. But this is true because

$$f(y + y'') \leq p(y' + y'') \leq p(y' + x - 0) + p(y'' - x_0)$$

by the subadditivity of p.

Step 2: The idea is to "iterate" Step 1 to get the general case. Let $\mathcal{P}$ be the collection of pairs (N, g) where N is a linear subspace such that $M \subseteq N \subseteq X$, $g : N \to \mathbb{R}$ is linear, and $g|_M = f$. We have the partial ordering $(N, g) \leq (N'.g')$ if $N \subseteq N'$ and $g'|_N = g$. If $((N_i, g_i))_i$ is a chain in $\mathcal{P}$, then $(\bigcup_i N_i, \bigcup_i g_i) \in \mathcal{P}$ is an upper bound for the chain. By Zorn's lemma, there is a maximal element $(N, g) \in \mathcal{P}$. We now must have $N = X$; otherwise, apply Step 1 to $N \subseteq N + \mathbb{R}x_1$ for some $x_1 \in X \setminus N$ to contradict the maximality of N. $\square$

Theorem 1.2 (complex Hahn-Banach). *Let $\mathbb{F} = \mathbb{C}$, let M be a linear subspace of X, and let p be a sublinear functional on X. If $f : M \to \mathbb{C}$ is such that $|f(x)| \leq p(x)$ for all $x \in M$, then there exists some linear $F : X \to \mathbb{C}$ such that $F|_M = f$ and $|F| \leq p$.*

Proof. Here is the sketch. Treat X as a real vector space. Then $g = \mathrm{Re}(f)$ is an $\mathbb{R}$-linear functional $M \to \mathbb{R}$. Extend g via the real Hahn-Banach theorem to get G on all of X. If $G : X \to \mathbb{R}$ is $\mathbb{R}$-linear, then $F(x) = F(x) - iG(ix)$ is $\mathbb{C}$-linear. Then $\|F\| = \|G\|$. $\square$

Here is the special case where p is a norm.

Corollary 1.1. *Let X be a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. If M is a linear subspace and $f \in M^*$, then there is an $F \in X^*$ such that $F|_M = f$ and $\|F\| = \|f\|$.*

1.3 Corollaries of Hahn-Banach

Corollary 1.2. *Let X be a normed space over $\mathbb{F}$, let $x_1, \ldots, x_n \in X$ be linearly independent, and let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. Then there is some $f \in X^*$ such that $f(x_i) = \alpha_i$ for all i.*

Proof. Define f by $f(x_i) = \alpha_i$ on $\mathrm{span}\{x_1, \ldots, x_n\}$. This is automatically bounded since this is a finite dimensional subspace. Now apply Hahn-Banach. $\square$

Corollary 1.3. *Let X be a normed space over $\mathbb{F}$, and let $x \in X$. Then* $\|x\| = \max\{|f(x)| : f \in X^*, \|f\| \leq 1\}$.

Proof. ($\geq$): This follows from the definition of the dual norm.

($\leq$): Apply the previous corollary with $x_1 = x$ and $\alpha_1 = \|x\|$. $\square$

Corollary 1.4. *Let X be a normed space over $\mathbb{F}$, let M be a non-dense linear subspace, and let $x \in X$. Then there is an $f \in X^*$ such that $f|_M = 0$, $\|f\| = 1$ and $f(x) = \operatorname{dist}(x, M)$.*

Proof. Consider the quotient map $Q : X \to X/M$. By Hahn-Banach, there exists an $f_0 \in (X/M)^*$ such that $\|f_0\| = 1$ and $f_0(x + M) = \operatorname{dist}(x, M)$. Let $f := f_0 \circ Q$. Then $f(y) = f_0(y + M)$ for all $y \in X$. $\square$

Corollary 1.5. *If X is a normed space and M is a linear subpsace, then*

$$\overline{M} = \bigcap_{\substack{f \in X^* \\ f|_M = 0}} \ker f.$$

Proof. ($\subseteq$): $\ker f \supseteq M$ for each element in the intersection, and each $\ker f$ is closed.

($\supseteq$): If $x \in X \setminus \overline{M}$, then take f from the previous corollary. Then $f(x) > 0$, so $x \notin \bigcap_f \ker f$. $\square$